

A Lyapunov functional and blow-up results for a class of perturbations for semilinear wave equations in the critical case

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Abstract

We consider in this paper some class of perturbation for the semilinear wave equation with critical (in the conformal transform sense) power nonlinearity. Working in the framework of similarity variables, we find a Lyapunov functional for the problem. Using a two-step argument based on interpolation and a critical Gagliardo-Nirenberg inequality, we show that the blow-up rate of any singular solution is given by the solution of the non perturbed associated ODE, namely $u'' = u^p$.

Keywords: Wave equation, finite time blow-up, blow-up rate, critical exponent, perturbations.

AMS classification : 35L05, 35L67, 35B20.

1 Introduction

This paper is devoted to the study of blow-up solutions for the following semilinear wave equation:

$$\begin{cases} \partial_{tt}u = \Delta u + |u|^{p-1}u + f(u) + g(\partial_t u), & (x, t) \in \mathbb{R}^N \times [0, T), \\ (u(x, 0), \partial_t u(x, 0)) = (u_0(x), u_1(x)) \in H_{loc,u}^1(\mathbb{R}^N) \times L_{loc,u}^2(\mathbb{R}^N), \end{cases} \quad (1.1)$$

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with critical power nonlinearity

$$p = p_c \equiv 1 + \frac{4}{N-1}, \quad \text{where } N \geq 2. \quad (1.2)$$

We assume that the functions f and g are locally Lipschitz-continuous satisfying the following conditions

$$(H_f) \quad |f(x)| \leq M(1 + |x|^q) \quad \text{with } (q < p, \quad M > 0),$$

$$(H_g) \quad |g(x)| \leq M(1 + |x|).$$

The spaces $L^2_{loc,u}(\mathbb{R}^N)$ and $H^1_{loc,u}(\mathbb{R}^N)$ are defined by

$$L^2_{loc,u}(\mathbb{R}^N) = \{u : \mathbb{R}^N \mapsto \mathbb{R} / \sup_{a \in \mathbb{R}^N} \left(\int_{|x-a| \leq 1} |u(x)|^2 dx \right) < +\infty\},$$

and

$$H^1_{loc,u}(\mathbb{R}^N) = \{u \in L^2_{loc,u}(\mathbb{R}^N), |\nabla u| \in L^2_{loc,u}(\mathbb{R}^N)\}.$$

The Cauchy problem of equation (1.1) is solved in $H^1_{loc,u} \times L^2_{loc,u}$. This follows from the finite speed of propagation and the wellposedness in $H^1 \times L^2$, valid whenever $1 < p < 1 + \frac{4}{N-2}$. The existence of blow-up solutions for the associated ordinary differential equation of (1.1) is a classical result. By using the finite speed of propagation, we conclude that there exists a blow-up solution $u(t)$ of (1.1) which depends non trivially on the space variable. In this paper, we consider a blow-up solution $u(t)$ of (1.1), we define (see for example Alinhac [1] and [2]) Γ as the graph of a function $x \mapsto T(x)$ such that the domain of definition of u is given by

$$D_u = \{(x, t) \mid t < T(x)\}.$$

The set D_u is called the maximal influence domain of u . Moreover, from the finite speed of propagation, T is a 1-Lipschitz function. Let \bar{T} be the minimum of $T(x)$ for all $x \in \mathbb{R}^N$. The time \bar{T} and the graph Γ are called (respectively) the blow-up time and the blow-up graph of u .

Let us first introduce the following non-degeneracy condition for Γ . If we introduce for all $x \in \mathbb{R}^N$, $t \leq T(x)$ and $\delta > 0$, the cone

$$C_{x,t,\delta} = \{(\xi, \tau) \neq (x, t) \mid 0 \leq \tau \leq t - \delta|\xi - x|\}, \quad (1.3)$$

then our non degeneracy condition is the following: x_0 is a non characteristic point if

$$\exists \delta_0 = \delta_0(x_0) \in (0, 1) \text{ such that } u \text{ is defined on } C_{x_0, T(x_0), \delta_0}. \quad (1.4)$$

We aim at studying the growth estimate of $u(t)$ near the space-time blow-up graph in the critical case (where $p = p_c$ satisfies (1.2)).

In the case $(f, g) \equiv (0, 0)$, equation (1.1) reduces to the semilinear wave equation:

$$\partial_{tt}u = \Delta u + |u|^{p-1}u, \quad (x, t) \in \mathbb{R}^N \times [0, \overline{T}). \quad (1.5)$$

Merle and Zaag in [9] (see also [7] and [8]) have proved, that if $1 < p \leq p_c = 1 + \frac{4}{N-1}$, if u is a solution of (1.5) with blow up graph $\Gamma : \{x \mapsto T(x)\}$, then for all $x_0 \in \mathbb{R}^N$ and $t \in [\frac{3}{4}T(x_0), T(x_0)]$, the growth estimate near the space-time blow-up graph satisfies

$$\begin{aligned} & (T(x_0) - t)^{\frac{2}{p-1}} \frac{\|u(t)\|_{L^2(B(x_0, \frac{T(x_0)-t}{2}))}}{(T(x_0) - t)^{\frac{N}{2}}} \\ & + (T(x_0) - t)^{\frac{2}{p-1}+1} \left(\frac{\|\partial_t u(t)\|_{L^2(B(x_0, \frac{T(x_0)-t}{2}))}}{(T(x_0) - t)^{\frac{N}{2}}} + \frac{\|\nabla u(t)\|_{L^2(B(x_0, \frac{T(x_0)-t}{2}))}}{(T(x_0) - t)^{\frac{N}{2}}} \right) \leq K, \end{aligned}$$

where the constant K depends only on N, p , and on an upper bound on $T(x_0)$, $\frac{1}{T(x_0)}$ and the initial data in $H_{loc,u}^1(\mathbb{R}^N) \times L_{loc,u}^2(\mathbb{R}^N)$. If in addition x_0 is non characteristic (in the sense (1.4)), then for all $t \in [\frac{3T(x_0)}{4}, T(x_0)]$,

$$\begin{aligned} 0 < \varepsilon_0(N, p) & \leq (T(x_0) - t)^{\frac{2}{p-1}} \frac{\|u(t)\|_{L^2(B(x_0, T(x_0)-t))}}{(T(x_0) - t)^{\frac{N}{2}}} \\ & + (T(x_0) - t)^{\frac{2}{p-1}+1} \left(\frac{\|\partial_t u(t)\|_{L^2(B(x_0, T(x_0)-t))}}{(T(x_0) - t)^{\frac{N}{2}}} + \frac{\|\nabla u(t)\|_{L^2(B(x_0, T(x_0)-t))}}{(T(x_0) - t)^{\frac{N}{2}}} \right) \leq K, \quad (1.6) \end{aligned}$$

where the constant K depends only on N, p , and on an upper bound on $T(x_0)$, $\frac{1}{T(x_0)}$, $\delta_0(x_0)$ and the initial data in $H_{loc,u}^1(\mathbb{R}^N) \times L_{loc,u}^2(\mathbb{R}^N)$.

Following this blow-up rate estimate, Merle and Zaag addressed the question of the asymptotic behavior of $u(x, t)$ near Γ in one space dimension.

More precisely, they proved in [10] and [11] that the set of non charecteristic points $\mathcal{R} \subset \mathbb{R}$ is non empty open and that $x \mapsto T(x)$ is of class C^1 on \mathcal{R} . They also described the blow-up profile of u near $(x_0, T(x_0))$ when $x_0 \in \mathcal{R}$.

In [12], they proved that $S = \mathbb{R} \setminus \mathcal{R}$ has an empty interior and that Γ is a corner of angle $\frac{\pi}{2}$ near any $x_0 \in S$. They also showed that $u(x, t)$ decomposes in a sum of decoupled solitons near $(x_0, T(x_0))$. They also gave examples of blow-up solutions with $S \neq \emptyset$.

In [6], we addressed the question of extending the results of Merle and Zaag [7], [8] and [9] to perturbed equations of the type (1.1). In [6], we could prove the statement (1.6) under some reasonable growth estimates an f and g in (1.1) however only when

$1 < p < p_c$. When $N \geq 2$ and $p = p_c$ our method in [6] breaks down. Let us briefly explain in the following how the method of [6] breaks down when $p = p_c$ justifying this way our new paper. In [6], we noticed that for the unperturbed equation (1.5) with $p \leq p_c$, Merle and Zaag [7], [8] and [9] crucially rely on the existence of a Lyapunov functional in similarity variables established by Antonini and Merle [3]. Following this idea, we introduce in [6] (for $p \leq p_c$) the similarity variables defined, for all $x_0 \in \mathbb{R}^N$, $0 < T_0 \leq T(x_0)$ by

$$y = \frac{x - x_0}{T_0 - t}, \quad s = -\log(T_0 - t), \quad u(x, t) = \frac{1}{(T_0 - t)^{\frac{2}{p-1}}} w_{x_0, T_0}(y, s). \quad (1.7)$$

From (1.1), the function w_{x_0, T_0} (we write w for simplicity) satisfies the following equation for all $y \in B \equiv B(0, 1)$ and $s \geq -\log T_0$:

$$\begin{aligned} \partial_{ss} w = & \frac{1}{\rho_\alpha} \operatorname{div}(\rho_\alpha \nabla w - \rho_\alpha (y \cdot \nabla w) y) - \frac{2p+2}{(p-1)^2} w + |w|^{p-1} w - \frac{p+3}{p-1} \partial_s w - 2y \cdot \nabla \partial_s w \\ & + e^{-\frac{2ps}{p-1}} f\left(e^{\frac{2s}{p-1}} w\right) + e^{-\frac{2ps}{p-1}} g\left(e^{\frac{(p+1)s}{p-1}} (\partial_s w + y \cdot \nabla w + \frac{2}{p-1} w)\right), \end{aligned} \quad (1.8)$$

where $\rho_\alpha = (1 - |y|^2)^\alpha$, with $\alpha = \alpha(N, p) = \frac{2}{p-1} - \frac{N-1}{2}$. In the new set of variables (y, s) , the behavior of u as $t \rightarrow T_0$ is equivalent to the behavior of w as $s \rightarrow +\infty$.

Following Antonini and Merle [3] and Merle and Zaag [8], we multiply equation (1.8) by $\rho_\alpha w_s$ and integrate over the unit ball B .

When $(f, g) \equiv (0, 0)$, we readily see from this calculation, as in [3] and [8], that E_α (defined below in (2.4)) is a Lyapunov functional in the sense that

$$\frac{d}{ds} E_\alpha(w) = -2\alpha \int_B (\partial_s w)^2 \frac{\rho_\alpha}{1 - |y|^2} dy, \quad (\text{if } p < p_c). \quad (1.9)$$

$$\frac{d}{ds} E_0(w) = - \int_{\partial B} (\partial_s w)^2 d\sigma, \quad (\text{if } p = p_c). \quad (1.10)$$

When $(f, g) \not\equiv (0, 0)$, (1.9) and (1.10) are perturbed by exponentially small terms with no sign. Our idea in [6] was to control these perturbation terms by the terms appearing in the definition of $E_\alpha(w)$ or its dissipation given in (1.9) and (1.10) where $(f, g) = (0, 0)$. Since these perturbations terms are supported in the whole unit ball B for $p \leq p_c$, our study works when $p < p_c$, because the dissipation in (1.9) is also supported in B . When $p = p_c$, the dissipation in (1.10) degenerates to the boundary and the method of [6] breaks down. That obstruction fully justifies our new paper, where we invent a new idea to get a Lyapunov functional for equation when $p = p_c$ (note that $\alpha(N, p_c) = 0$). Let us explain in the following our new idea. In fact, our strategy relies on two steps:

Part 1: A rough estimate. As we said above, if we study $E_0(w)$, the functional of the unperturbed case $(f, g) \equiv (0, 0)$, then we get exponentially small terms that we can't control (unlike the subcritical case $p < p_c$). In other words, the study of $E_0(w)$ cannot be extended from the unperturbed case to the general case. Fortunately, we noticed that the study of $E_\eta(w)$ where $\eta > 0$ can be extended from the case $(f, g) \equiv (0, 0)$ to the case $(f, g) \not\equiv (0, 0)$. It happens that in the former case, $E_\eta(w)$ has a very bad bound, in the sense that

$$E_\eta(w(s)) \leq Ce^{\beta\eta s},$$

for some $\beta = \beta(p)$ and

$$\|w(s)\|_{H^1(B)} + \|\partial_s w(s)\|_{L^2(B)} \leq Ce^{\beta\eta s}. \quad (1.11)$$

This behavior is conserved when $(f, g) \not\equiv (0, 0)$.

Part 2: The sharp estimate. Now, we go back to $E_0(w)$, which is the "good" functional for $(f, g) \equiv (0, 0)$, in the sense that it is bounded. When $(f, g) \not\equiv (0, 0)$, if we study $E_0(w)$, the exponentially small terms are no longer a problem for us, thanks the rough estimate of (1.11), provided that we fix η small enough.

The equation (1.8) will be studied in the space \mathcal{H}

$$\mathcal{H} = \left\{ (w_1, w_2), \left| \int_B \left(w_2^2 + |\nabla w_1|^2(1 - |y|^2) + w_1^2 \right) dy < +\infty \right. \right\}.$$

In the whole paper, we denote $F(u) = \int_0^u f(v)dv$ and we assume that (1.2) holds.

In the case $(f, g) \equiv (0, 0)$, Merle and Zaag [8] proved that

$$E_0(w) = \int_B \left(\frac{1}{2}(\partial_s w)^2 + \frac{1}{2}|\nabla w|^2 - \frac{1}{2}(y \cdot \nabla w)^2 + \frac{p+1}{(p-1)^2}w^2 - \frac{1}{p+1}|w|^{p+1} \right) dy, \quad (1.12)$$

is a Lyapunov functional for equation (1.8). When $(f, g) \not\equiv (0, 0)$, we introduce

$$H(w) = E(w) + \sigma e^{-\gamma s}, \quad (1.13)$$

where σ is a sufficiently large constant that will be determined later,

$$\begin{aligned} E(w) &= E_0(w) + I_0(w) \quad \text{and} \quad I_0(w) = -e^{-\frac{2(p+1)s}{p-1}} \int_B F(e^{\frac{2s}{p-1}} w) dy, \\ \text{with} \quad \gamma &= \min\left(\frac{1}{2}, \frac{p-q}{p-1}\right) > 0. \end{aligned} \quad (1.14)$$

Here we announce our main result.

THEOREM 1.1 (*Existence of a Lyapunov functional for equation (1.8)*)

Consider u a solution of (1.1) with blow-up graph $\Gamma : \{x \mapsto T(x)\}$ and x_0 is a non characteristic point. Then there exists $t_0(x_0) \in [0, T(x_0))$ such that, for all $T_0 \in (t_0(x_0), T_0(x_0)]$, for all $s \geq -\log(T_0 - t_0(x_0))$, we have

$$H(w(s+10)) - H(w(s)) \leq - \int_s^{s+10} \int_{\partial B} (\partial_s w)^2(\sigma, \tau) d\sigma d\tau, \quad (1.15)$$

where $w = w_{x_0, T_0}$ is defined in (1.7).

Remark 1.1.

- Since we crucially need a covering technique in our argument (see Appendix A), in fact, we need a uniform version for x near x_0 (see theorem 1.1' page 18 below).
- One may wonder why we take only sublinear perturbations in $\partial_t u$ (see hypothesis (H_g)). It happens that any superlinear terms in $(\partial_t u)$ generates in similarity variables L^r norms of $\partial_s w$ and ∇w , where $r > 2$, hence, non controllable by the terms in the Lyapunov functional $E_0(w)$ (1.12) of the non perturbed equation (1.5).

The existence of this Lyapunov functional (and a blow-up criterion for equation (1.8) based in H) are a crucial step in the derivation of the blow-up rate for equation (1.1). Indeed, with the functional H and some more work, we are able to adapt the analysis performed in [9] for equation (1.5) and get the following result:

THEOREM 1.2 (*Blow-up rate for equation (1.1)*)

There exist $\varepsilon_0 > 0$ such that if u is a solution of (1.1) with blow-up graph $\Gamma : \{x \mapsto T(x)\}$ and x_0 is a non characteristic point, then there exist $\widehat{S}_2 > 0$ such that

i) For all $s \geq \widehat{s}_2(x_0) = \max(\widehat{S}_2, -\log \frac{T(x_0)}{4})$,

$$0 < \varepsilon_0 \leq \|w_{x_0, T(x_0)}(s)\|_{H^1(B)} + \|\partial_s w_{x_0, T(x_0)}(s)\|_{L^2(B)} \leq K,$$

where $w_{x_0, T(x_0)}$ is defined in (1.7) and B is the unit ball of \mathbb{R}^N .

ii) For all $t \in [t_2(x_0), T(x_0))$, where $t_2(x_0) = T(x_0) - e^{-\widehat{s}_2(x_0)}$, we have

$$\begin{aligned} 0 < \varepsilon_0 \leq & (T(x_0) - t)^{\frac{2}{p-1}} \frac{\|u(t)\|_{L^2(B(x_0, T(x_0)-t))}}{(T(x_0) - t)^{\frac{N}{2}}} \\ & + (T(x_0) - t)^{\frac{2}{p-1}+1} \left(\frac{\|\partial_t u(t)\|_{L^2(B(x_0, T(x_0)-t))}}{(T(x_0) - t)^{\frac{N}{2}}} + \frac{\|\nabla u(t)\|_{L^2(B(x_0, T(x_0)-t))}}{(T(x_0) - t)^{\frac{N}{2}}} \right) \leq K, \end{aligned}$$

where $K = K(\widehat{s}_2(x_0), \|(u(t_2(x_0)), \partial_t u(t_2(x_0)))\|_{H^1 \times L^2(B(x_0, \frac{e^{-\widehat{s}_2(x_0)}}{\delta_0(x_0)}))})$ and $\delta_0(x_0) \in (0, 1)$ is defined in (1.4).

Remark 1.2. With this blow-up rate, one can ask whether the results proved by Merle and Zaag for the non perturbed problem in [10] [11] [12], hold for equation (1.1) (blow-up, profile, regularity of the blow-up graph, existence of characteristic points, etc...). We believe that it is the case, however, the proof will be highly technical, with no interesting ideas (in particular, equation (1.1) is not conserved under the Lorentz transform, which is crucial in [10] [11] [12], and lots of minor term will appear in the analysis). Once again, we believe that the key point in the analysis of blow-up for equation (1.1) is the derivation of a Lyapunov functional in similarity variables, which is the object of our paper.

As in the particular case where $(f, g) \equiv (0, 0)$, the proof of theorem 1.2 relies on four ideas (the existence of a Lyapunov functional, interpolation in Sobolev spaces, some critical Gagliardo-Nirenberg estimates and a covering technique adapted to the geometric shape of the blow-up surface). It happens that adapting the proof of [9] given in the non perturbed case (1.5) is straightforward, except for a key argument, where we bound the L^{p+1} space-time norm of w . Therefore, we only present that argument, and refer to [6], [7] and [9] for the rest of the proof.

This paper is organized as follows: In section 2, we obtain a rough control on space-time of the solution w . Using this result, we proved in section 3, that the "natural" functional is a Lyapunov functional for equation (1.8). Thus, we get theorem 1.1. Finally, applying this last theorem and method used in [9], we easily prove theorem 1.2.

2 A rough estimate

In this section, we prove a rough version of *i*) of theorem 1.2, where we obtain an exponentially growing bound on time averages of the $H^1 \times L^2(B)$ norm of $(w, \partial_s w)$. Consider u a solution of (1.1) with blow-up graph $\Gamma : \{x \mapsto T(x)\}$ and x_0 is a non characteristic point. More precisely, this is the aim of this section.

PROPOSITION 2.1 *For all $\eta \in (0, 1)$, there exists $t_0(x_0) \in [0, T(x_0))$ such that, for all $T_0 \in (t_0(x_0), T(x_0)]$, for all $s \geq -\log(T_0 - t_0(x_0))$ and $x \in \mathbb{R}^N$ where $|x - x_0| \leq \frac{e^{-s}}{\delta_0(x_0)}$, we have*

$$\begin{aligned} \int_s^{s+10} \int_B (\partial_s w)^2(y, \tau) dy d\tau + \int_s^{s+10} \int_B |w(y, \tau)|^{p+1} dy d\tau \\ + \int_s^{s+10} \int_B |\nabla w(y, \tau)|^2 dy d\tau \leq K_1 e^{\frac{\eta(p+3)s}{2}}, \end{aligned} \quad (2.1)$$

where $w = w_{x, T^*(x)}$ is defined in (1.7) with

$$T^*(x) = T_0 - \delta_0(x_0)(x - x_0), \quad (2.2)$$

$K_1 = K_1(\eta, T(x_0) - t_0(x_0), \|(u(t_0(x_0)), \partial_t u(t_0(x_0)))\|_{H^1 \times L^2(B(x_0, \frac{T(x_0) - t_0(x_0)}{\delta_0(x_0)}))})$ and $\delta_0(x_0) \in (0, 1)$ is defined in (1.4).

2.1 Rough energy estimates

Consider $T_0 \in (0, T(x_0)]$, for all $x \in \mathbb{R}^N$ is such that $|x - x_0| \leq \frac{T_0}{\delta_0(x_0)}$, where $\delta(x_0)$ is defined in (1.4), then we write w instead of $w_{x, T^*(x)}$ defined in (1.7) with $T^*(x)$ given in (2.2). Let $\eta \in (0, 1)$ and write the equation (1.8) satisfied by w in the form

$$\begin{aligned} \partial_{ss} w &= \frac{1}{\rho_\eta} \operatorname{div}(\rho_\eta \nabla w - \rho_\eta (y \cdot \nabla w) y) + 2\eta y \cdot \nabla w - \frac{2p+2}{(p-1)^2} w + |w|^{p-1} w \\ &\quad - \frac{p+3}{p-1} \partial_s w - 2y \cdot \nabla \partial_s w + e^{-\frac{2ps}{p-1}} f\left(e^{\frac{2s}{p-1}} w\right) \\ &\quad + e^{-\frac{2ps}{p-1}} g\left(e^{\frac{(p+1)s}{p-1}} (\partial_s w + y \cdot \nabla w + \frac{2}{p-1} w)\right), \quad \forall y \in B \text{ and } s \geq -\log T^*(x), \end{aligned} \quad (2.3)$$

where $\rho_\eta = (1 - |y|^2)^\eta$. We denote by C a constant which depends on η .

To control the norm of $(w(s), \partial_s w(s)) \in \mathcal{H}$, we first introduce the following functionals

$$\begin{aligned} E_\eta(w) &= \int_B \left(\frac{1}{2} (\partial_s w)^2 + \frac{1}{2} |\nabla w|^2 - \frac{1}{2} (y \cdot \nabla w)^2 + \frac{p+1}{(p-1)^2} w^2 - \frac{1}{p+1} |w|^{p+1} \right) \rho_\eta dy, \\ I_\eta(w) &= -e^{-\frac{2(p+1)s}{p-1}} \int_B F(e^{\frac{2s}{p-1}} w) \rho_\eta dy, \\ J_\eta(w) &= -\eta \int_B w \partial_s w \rho_\eta dy + \frac{N\eta}{2} \int_B w^2 \rho_\eta dy, \\ H_\eta(w) &= E_\eta(w) + I_\eta(w) + J_\eta(w), \\ G_\eta(w) &= H_\eta(w) e^{-\frac{\eta(p+3)s}{2}} + \theta e^{-\frac{\eta(p+3)s}{2}}, \end{aligned} \quad (2.4)$$

where $\theta = \theta(\eta)$ is a sufficiently large constant that will be determined later. In this subsection, we prove that $G_\eta(w)$ is decreasing in time, which will give the rough (ie exponentially fast) estimate for $E_\eta(w)$ and $\|(w, \partial_s w)\|_{H^1(B) \times L^2(B)}$.

Now we state two lemmas which are crucial for the proof. We begin with bounding the time derivative of $E_\eta(w)$ in the following lemma.

LEMMA 2.2 *For all $\eta \in (0, 1)$, we have the following inequality, for all $s \geq \max(-\log T^*(x), 0)$,*

$$\frac{d}{ds} (E_\eta(w) + I_\eta(w)) \leq -2\eta \int_B (\partial_s w)^2 \frac{|y|^2 \rho_\eta}{1 - |y|^2} dy + 2\eta \int_B \partial_s w (y \cdot \nabla w) \rho_\eta dy + \Sigma_0(s), \quad (2.5)$$

where $\Sigma_0(s)$ satisfies

$$\begin{aligned}\Sigma_0(s) &\leq Ce^{-2\gamma s} + Ce^{-2\gamma s} \int_B |\nabla w|^2 (1 - |y|^2) \rho_\eta dy + Ce^{-2\gamma s} \int_B w^2 \rho_\eta dy \\ &\quad + Ce^{-2\gamma s} \int_B (\partial_s w)^2 \frac{\rho_\eta}{1 - |y|^2} dy + Ce^{-2\gamma s} \int_B |w|^{p+1} \rho_\eta dy,\end{aligned}\quad (2.6)$$

with $\gamma = \min(\frac{1}{2}, \frac{p-q}{p-1}) > 0$.

Proof. Multipling (2.3) by $\rho_\eta \partial_s w$ and integrating over the ball B , we obtain for all $s \geq -\log T^*(x)$,

$$\begin{aligned}\frac{d}{ds}(E_\eta(w) + I_\eta(w)) &= -2 \int_B (\partial_s w)(y \cdot \nabla \partial_s w) \rho_\eta dy - \frac{p+3}{p-1} \int_B (\partial_s w)^2 \rho_\eta dy \\ &\quad + 2\eta \int_B (\partial_s w)(y \cdot \nabla w) \rho_\eta dy \\ &\quad + \underbrace{\frac{2(p+1)}{p-1} e^{-\frac{2(p+1)s}{p-1}} \int_B F\left(e^{\frac{2s}{p-1}} w\right) \rho_\eta dy}_{\Sigma_0^1(s)} \\ &\quad + \underbrace{\frac{2}{p-1} e^{-\frac{2ps}{p-1}} \int_B f\left(e^{\frac{2s}{p-1}} w\right) w \rho_\eta dy}_{\Sigma_0^2(s)} \\ &\quad + \underbrace{e^{-\frac{2ps}{p-1}} \int_B g\left(e^{\frac{(p+1)s}{p-1}} (\partial_s w + y \cdot \nabla w + \frac{2}{p-1} w)\right) \partial_s w \rho_\eta dy}_{\Sigma_0^3(s)}.\end{aligned}\quad (2.7)$$

Then by integrating by parts, we have

$$\begin{aligned}\frac{d}{ds}(E_\eta(w) + I_\eta(w)) &= -2\eta \int_B (\partial_s w)^2 \frac{|y|^2 \rho_\eta}{1 - |y|^2} dy + 2\eta \int_B (\partial_s w)(y \cdot \nabla w) \rho_\eta dy \\ &\quad + \Sigma_0^1(s) + \Sigma_0^2(s) + \Sigma_0^3(s).\end{aligned}\quad (2.8)$$

Using the fact that $|F(x)| + |xf(x)| \leq C(1 + |x|^{p+1})$, we obtain that for all $s \geq \max(-\log T^*(x), 0)$,

$$|\Sigma_0^1(s)| + |\Sigma_0^2(s)| \leq Ce^{-2\gamma s} + Ce^{-2\gamma s} \int_B |w|^{p+1} \rho_\eta dy, \quad (2.9)$$

on the one hand. On the other hand, since $|g(x)| \leq M(1 + |x|)$, we write that for all $s \geq \max(-\log T^*(x), 0)$,

$$\begin{aligned}|\Sigma_0^3(s)| &\leq Ce^{-2\gamma s} \int_B (\partial_s w)^2 \rho_\eta dy + Ce^{-2\gamma s} \int_B |y \cdot \nabla w| |\partial_s w| \rho_\eta dy \\ &\quad + Ce^{-2\gamma s} \int_B |w \partial_s w| \rho_\eta dy + Ce^{-2\gamma s} \int_B |\partial_s w| \rho_\eta dy.\end{aligned}$$

By exploiting the inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$, we conclude that for all $s \geq \max(-\log T^*(x), 0)$,

$$\begin{aligned} |\Sigma_0^3(s)| &\leq Ce^{-2\gamma s} \int_B (\partial_s w)^2 \frac{\rho_\eta}{1-|y|^2} dy + Ce^{-2\gamma s} \int_B w^2 \rho_\eta dy \\ &\quad + Ce^{-2\gamma s} \int_B |\nabla w|^2 (1-|y|^2) \rho_\eta dy + Ce^{-2\gamma s}. \end{aligned} \quad (2.10)$$

Then, by using (2.8), (2.9) and (2.10), we have for all $s \geq \max(-\log T^*(x), 0)$, the estimates (2.5) and (2.6) hold. This concludes the proof of lemma 2.2. ■

We are now going to prove the following estimate for the functional J_η :

LEMMA 2.3 *For all $\eta \in (0, 1)$, J_η satisfies the following inequality, for all $s \geq \max(-\log T^*(x), 0)$*

$$\begin{aligned} \frac{d}{ds} J_\eta(w) &\leq \frac{32\eta}{p+15} \int_B (\partial_s w)^2 \frac{\rho_\eta}{1-|y|^2} dy - 2\eta \int_B \partial_s w (y \cdot \nabla w) \rho_\eta dy + \frac{\eta(p+3)}{2} H_\eta \\ &\quad - \frac{\eta(p+15)}{8} \int_B (\partial_s w)^2 \rho_\eta dy - \frac{\eta(p-1)}{2(p+1)} \int_B |w|^{p+1} \rho_\eta dy \\ &\quad - \frac{\eta(p-1)}{8} \int_B |\nabla w|^2 (1-|y|^2) \rho_\eta dy + \Sigma_1(s), \end{aligned} \quad (2.11)$$

where $\Sigma_1(s)$ satisfies

$$\begin{aligned} \Sigma_1(s) &\leq Ce^{-2\gamma s} \int_B (\partial_s w)^2 \frac{\rho_\eta}{1-|y|^2} dy + Ce^{-2\gamma s} \int_B |w|^{p+1} \rho_\eta dy \\ &\quad + Ce^{-2\gamma s} \int_B |\nabla w|^2 (1-|y|^2) \rho_\eta dy + C \int_B w^2 \rho_\eta dy + Ce^{-2\gamma s}, \end{aligned} \quad (2.12)$$

with $\gamma = \min(\frac{1}{2}, \frac{p-q}{p-1})$.

Proof: Note that J_η is a differentiable function for all $s \geq -\log T^*(x)$ and that

$$\frac{d}{ds} J_\eta(w) = -\eta \int_B (\partial_s w)^2 \rho_\eta dy - \eta \int_B w \partial_{ss} w \rho_\eta dy + N\eta \int_B w \partial_s w \rho_\eta dy.$$

By using equation (2.3) and integrating by parts, we have

$$\begin{aligned}
\frac{d}{ds} J_\eta(w) = & -\eta \int_B (\partial_s w)^2 \rho_\eta dy + \eta \int_B (|\nabla w|^2 - (y \cdot \nabla w)^2) \rho_\eta dy - \eta \int_B |w|^{p+1} \rho_\eta dy \\
& - 2\eta \int_B \partial_s w (y \cdot \nabla w) \rho_\eta dy + \eta \left(\frac{2p+2}{(p-1)^2} + \eta N \right) \int_B w^2 \rho_\eta dy \\
& + 4\eta^2 \underbrace{\int_B w \partial_s w \frac{|y|^2 \rho_\eta}{1-|y|^2} dy}_{\Sigma_1^1(s)} - 2\eta^3 \underbrace{\int_B w^2 \frac{|y|^2 \rho_\eta}{1-|y|^2} dy}_{\Sigma_1^2(s)} \\
& - \underbrace{\eta e^{-\frac{2ps}{p-1}} \int_B w f \left(e^{\frac{2s}{p-1}} w \right) \rho_\eta dy}_{\Sigma_1^3(s)} \\
& - \underbrace{\eta e^{-\frac{2ps}{p-1}} \int_B w g \left(e^{\frac{(p+1)s}{p-1}} (\partial_s w + y \cdot \nabla w + \frac{2}{p-1} w) \right) \rho_\eta dy}_{\Sigma_1^4(s)}. \tag{2.13}
\end{aligned}$$

By combining (2.4) and (2.13), we write

$$\begin{aligned}
\frac{d}{ds} J_\eta(w) = & -2\eta \int_B \partial_s w (y \cdot \nabla w) \rho_\eta dy + \frac{\eta(p+3)}{2} H_\eta - \frac{\eta(p+7)}{4} \int_B (\partial_s w)^2 \rho_\eta dy \\
& - \frac{\eta(p-1)}{2(p+1)} \int_B |w|^{p+1} \rho_\eta dy - \frac{\eta(p-1)}{4} \int_B (|\nabla w|^2 - (y \cdot \nabla w)^2) \rho_\eta dy \tag{2.14} \\
& - \eta \left(\frac{p+1}{2(p-1)} + \frac{\eta N(p-1)}{4} \right) \int_B w^2 \rho_\eta dy + \underbrace{\frac{\eta^2(p+3)}{2} \int_B w \partial_s w \rho_\eta dy}_{\Sigma_1^5(s)} \\
& + \underbrace{\frac{\eta(p+3)}{2} e^{-\frac{2(p+1)s}{p-1}} \int_B F(e^{\frac{2}{p-1}s} w) \rho_\eta dy}_{\Sigma_1^6(s)} + \Sigma_1^1(s) + \Sigma_1^2(s) + \Sigma_1^3(s) + \Sigma_1^4(s).
\end{aligned}$$

We now study each of the last six terms.

By the Cauchy-Schwartz inequality we write, for all $\mu \in (0, 1)$

$$\Sigma_1^1(s) \leq 2\eta(1-\mu) \int_B (\partial_s w)^2 \frac{\rho_\eta}{1-|y|^2} dy + \frac{2\eta^3}{1-\mu} \int_B w^2 \frac{|y|^2 \rho_\eta}{1-|y|^2} dy.$$

We use the expression of $\Sigma_1^2(s)$ to obtain, for all $\mu \in (0, 1)$

$$\Sigma_1^1(s) + \Sigma_1^2(s) \leq 2\eta(1-\mu) \int_B (\partial_s w)^2 \frac{\rho_\eta}{1-|y|^2} dy + \frac{2\eta^3 \mu}{1-\mu} \int_B w^2 \frac{|y|^2 \rho_\eta}{1-|y|^2} dy. \tag{2.15}$$

Since we have the following Hardy type inequality for any $w \in H_{loc,u}^1(\mathbb{R}^N)$ (see appendix B in [7] for details):

$$\int_B w^2 \frac{|y|^2 \rho_\eta}{1-|y|^2} dy \leq \frac{1}{\eta^2} \int_B |\nabla w|^2 (1-|y|^2) \rho_\eta dy + \frac{N}{\eta} \int_B w^2 \rho_\eta dy, \tag{2.16}$$

from (2.15) and (2.16) and if we choose $\mu = \frac{p-1}{p+15}$, we conclude that

$$\begin{aligned}\Sigma_1^1(s) + \Sigma_1^2(s) &\leq \frac{32\eta}{p+15} \int_B (\partial_s w)^2 \frac{\rho_\eta}{1-|y|^2} dy + \frac{\eta^2 N(p-1)}{8} \int_B w^2 \rho_\eta dy \\ &\quad + \frac{\eta(p-1)}{8} \int_B |\nabla w|^2 (1-|y|^2) \rho_\eta dy.\end{aligned}\quad (2.17)$$

By exploiting the fact that $|F(x)| + |xf(x)| \leq C(1 + |x|^{p+1})$, we obtain for all $s \geq \max(-\log T^*(x), 0)$

$$\Sigma_1^3(s) + \Sigma_1^6(s) \leq Ce^{-2\gamma s} + Ce^{-2\gamma s} \int_B |w|^{p+1} \rho_\eta dy. \quad (2.18)$$

In a similar way, by using the fact that $|g(x)| \leq M(1 + |x|)$, we write for all $s \geq \max(-\log T^*(x), 0)$

$$\begin{aligned}\Sigma_1^4(s) &\leq Ce^{-2\gamma s} \int_B w^2 \rho_\eta dy + Ce^{-2\gamma s} \int_B |y \cdot \nabla w| |w| \rho_\eta dy \\ &\quad + Ce^{-2\gamma s} \int_B (\partial_s w)^2 \rho_\eta dy + Ce^{-2\gamma s}.\end{aligned}$$

By using (2.16), we get

$$\begin{aligned}\Sigma_1^4(s) &\leq Ce^{-2\gamma s} \int_B (\partial_s w)^2 \rho_\eta dy + Ce^{-2\gamma s} \int_B |\nabla w|^2 (1-|y|^2) \rho_\eta dy \\ &\quad + Ce^{-2\gamma s} \int_B w^2 \rho_\eta dy + Ce^{-2\gamma s}.\end{aligned}\quad (2.19)$$

To estimate $\Sigma_1^5(s)$, we use the Cauchy-Schwartz inequality and we get

$$\Sigma_1^5(s) \leq \frac{(p-1)\eta}{8} \int_B (\partial_s w)^2 \rho_\eta dy + C \int_B w^2 \rho_\eta dy. \quad (2.20)$$

Since $|y \cdot \nabla w| \leq |y| |\nabla w|$, it follows that

$$\int_B |\nabla w|^2 (1-|y|^2) \rho_\eta dy \leq \int_B (|\nabla w|^2 - (y \cdot \nabla w)^2) \rho_\eta dy. \quad (2.21)$$

Finally, by using (2.14), (2.17), (2.18), (2.19), (2.20) and (2.21), we have easily the estimate (2.11) and (2.12). This concludes the proof of lemma 2.3. \blacksquare

From lemmas 2.2 and 2.3, we are in a position to prove the following proposition

PROPOSITION 2.4 (*Existence of a decreasing functional for equation (1.8)*)

For all $\eta \in (0, 1)$, there exists $S_0 > 0$ such that, G_η defined in (2.4) satisfies the following inequality, for all $s_2 > s_1 \geq \max(-\log T^*(x), S_0)$,

$$\begin{aligned} G_\eta(w(s_2)) - G_\eta(w(s_1)) &\leq -\frac{\eta(p-1)}{p+15} \int_{s_1}^{s_2} e^{-\frac{\eta(p+3)s}{2}} \int_B (\partial_s w)^2 \frac{\rho_\eta}{1-|y|^2} dy ds \\ &\quad - \frac{\eta(p-1)}{8(p+1)} \int_{s_1}^{s_2} e^{-\frac{\eta(p+3)s}{2}} \int_B |w|^{p+1} \rho_\eta dy ds \\ &\quad - \frac{\eta(p-1)}{16} \int_{s_1}^{s_2} e^{-\frac{\eta(p+3)s}{2}} \int_B |\nabla w|^2 (1-|y|^2) \rho_\eta dy ds. \end{aligned} \quad (2.22)$$

Proof: From lemmas 2.2 and 2.3, we obtain for all $s \geq \max(-\log T^*(x), 0)$,

$$\begin{aligned} \frac{d}{ds} H_\eta(w) &\leq \frac{\eta(p+3)}{2} H_\eta(w) - \left(\frac{2\eta(p-1)}{p+15} - C e^{-2\gamma s} \right) \int_B (\partial_s w)^2 \frac{\rho_\eta}{1-|y|^2} dy \\ &\quad - \left(\frac{\eta(p-1)}{2(p+1)} - C e^{-2\gamma s} \right) \int_B |w|^{p+1} \rho_\eta dy \\ &\quad - \left(\frac{\eta(p-1)}{8} - C e^{-2\gamma s} \right) \int_B |\nabla w|^2 (1-|y|^2) \rho_\eta dy \\ &\quad + C \int_B w^2 \rho_\eta dy + C e^{-2\gamma s}. \end{aligned} \quad (2.23)$$

We now choose S_0 large enough ($S_0 \geq 0$), so that for all $s \geq S_0$, we have

$$\frac{\eta(p-1)}{p+15} - C e^{-2\gamma s} \geq 0, \quad \frac{\eta(p-1)}{4(p+1)} - C e^{-2\gamma s} \geq 0, \quad \frac{\eta(p-1)}{16} - C e^{-2\gamma s} \geq 0.$$

Then, we deduce that, for all $s \geq \max(-\log T^*(x), S_0)$, we have

$$\begin{aligned} \frac{d}{ds} H_\eta(w) &\leq \frac{\eta(p+3)}{2} H_\eta(w) - \frac{\eta(p-1)}{p+15} \int_B (\partial_s w)^2 \frac{\rho_\eta}{1-|y|^2} dy - \frac{\eta(p-1)}{4(p+1)} \int_B |w|^{p+1} \rho_\eta dy \\ &\quad - \frac{\eta(p-1)}{16} \int_B |\nabla w|^2 (1-|y|^2) \rho_\eta dy + C \int_B w^2 \rho_\eta dy + C e^{-2\gamma s}. \end{aligned} \quad (2.24)$$

By combining (2.24) and the following Jensen's inequality

$$C \int_B w^2 \rho_\eta dy \leq \frac{\eta(p-1)}{8(p+1)} \int_B |w|^{p+1} \rho_\eta dy + C, \quad (2.25)$$

we obtain, for all $s \geq \max(-\log T^*(x), S_0)$,

$$\begin{aligned} \frac{d}{ds} H_\eta(w) &\leq \frac{\eta(p+3)}{2} H_\eta(w) - \frac{\eta(p-1)}{p+15} \int_B (\partial_s w)^2 \frac{\rho_\eta}{1-|y|^2} dy - \frac{\eta(p-1)}{8(p+1)} \int_B |w|^{p+1} \rho_\eta dy \\ &\quad - \frac{\eta(p-1)}{16} \int_B |\nabla w|^2 (1-|y|^2) \rho_\eta dy + C. \end{aligned} \quad (2.26)$$

Finally, by (2.26), we prove easily that the function G_η satisfies, for all $s \geq \max(-\log T^*(x), S_0)$,

$$\begin{aligned} \frac{d}{ds} G_\eta(w) \leq & -\frac{\eta(p-1)}{p+15} e^{-\frac{\eta(p+3)s}{2}} \int_B (\partial_s w)^2 \frac{\rho_\eta}{1-|y|^2} dy - \frac{\eta(p-1)}{8(p+1)} e^{-\frac{\eta(p+3)s}{2}} \int_B |w|^{p+1} \rho_\eta dy \\ & - \frac{\eta(p-1)}{16} e^{-\frac{\eta(p+3)s}{2}} \int_B |\nabla w|^2 (1-|y|^2) \rho_\eta dy + (C - \frac{\eta\theta(p+3)}{2}) e^{-\frac{\eta(p+3)s}{2}}. \end{aligned}$$

We now choose $\theta = \theta(\eta)$ large enough, so we have $C - \frac{\eta\theta(p+3)}{2} \leq 0$ and then

$$\begin{aligned} \frac{d}{ds} G_\eta(w) \leq & -\frac{\eta(p-1)}{p+15} e^{-\frac{\eta(p+3)s}{2}} \int_B (\partial_s w)^2 \frac{\rho_\eta}{1-|y|^2} dy - \frac{\eta(p-1)}{8(p+1)} e^{-\frac{\eta(p+3)s}{2}} \int_B |w|^{p+1} \rho_\eta dy \\ & - \frac{\eta(p-1)}{16} e^{-\frac{\eta(p+3)s}{2}} \int_B |\nabla w|^2 (1-|y|^2) \rho_\eta dy. \end{aligned} \quad (2.27)$$

Now (2.22) is a direct consequence of inequality (2.27). This concludes the proof of proposition 2.4. \blacksquare

2.2 Proof of proposition 2.1

We now claim the following lemma:

LEMMA 2.5 *For all $\eta \in (0, 1)$, there exists $S_1 \geq S_0$ such that, if $G_\eta(w(s_1)) < 0$ for some $s_1 \geq \max(-\log T^*(x), S_1)$, then w blows up in some finite time $S > s_1$.*

Proof: The argument is the same as in the corresponding part in [6]. Let us remark that our proof strongly relies on the fact that $p < 1 + \frac{4}{N-2}$ which is implied by the fact that $p = p_c = 1 + \frac{4}{N-1}$. \blacksquare

We define the following time:

$$t_0(x_0) = \max(T(x_0) - e^{-S_1}, 0). \quad (2.28)$$

Since $\eta \in (0, 1)$, by combining proposition 2.4 and lemma 2.5, we get the following bounds:

COROLLARY 2.6 *(Estimates on H_η) For all $\eta \in (0, 1)$, there exists $t_0(x_0) \in [0, T(x_0))$ such that, for all $T_0 \in (t_0(x_0), T(x_0)]$, for all $s \geq -\log(T_0 - t_0(x_0))$ and $x \in \mathbb{R}^N$ where $|x - x_0| \leq \frac{e^{-s}}{\delta_0(x_0)}$, we have*

$$\begin{aligned} (i) \quad & -C \leq H_\eta(w(s)) \leq \left(\theta + H_\eta(w(\tilde{s}_0)) \right) e^{\frac{\eta(p+3)s}{2}}, \\ & \int_s^{s+10} \int_B (\partial_s w)^2(y, \tau) \frac{\rho_\eta}{1-|y|^2} dy d\tau \leq C \left(\theta + H_\eta(w(\tilde{s}_0)) \right) e^{\frac{\eta(p+3)s}{2}}, \end{aligned} \quad (2.29)$$

$$\begin{aligned} \int_s^{s+10} \int_B |w(y, \tau)|^{p+1} \rho_\eta dy d\tau + \int_s^{s+10} \int_B |\nabla w(y, \tau)|^2 (1 - |y|^2) \rho_\eta dy d\tau \\ \leq C(\theta + H_\eta(w(\tilde{s}_0))) e^{\frac{\eta(p+3)s}{2}}. \end{aligned} \quad (2.30)$$

(ii)

$$\int_s^{s+10} \int_B (\partial_s w)^2(y, \tau) dy d\tau \leq C(\theta + CH_\eta(w(\tilde{s}_0))) e^{\frac{\eta(p+3)s}{2}}, \quad (2.31)$$

$$\begin{aligned} \int_s^{s+10} \int_{B_{\frac{1}{2}}} |w(y, \tau)|^{p+1} dy d\tau + \int_s^{s+10} \int_{B_{\frac{1}{2}}} |\nabla w(y, \tau)|^2 dy d\tau \\ \leq C(\theta + H_\eta(w(\tilde{s}_0))) e^{\frac{\eta(p+3)s}{2}}, \end{aligned} \quad (2.32)$$

where $w = w_{x, T^*(x)}$ is defined in (1.7), $T^*(x)$ is defined in (2.2) and $\tilde{s}_0 = -\log(T^*(x) - t_0(x_0))$.

Remark 2.2. By using the definition of (1.7) of $w_{x, T^*(x)} = w$, we write easily

$$C\theta + CH_\eta(w(\tilde{s}_0)) \leq K_0,$$

where $K_0 = K_0(\eta, T(x_0) - t_0(x_0), \|(u(t_0(x_0)), \partial_t u(t_0(x_0)))\|_{H^1 \times L^2(B(x_0, \frac{T(x_0) - t_0(x_0)}{\delta_0(x_0)}))})$ and $\delta_0(x_0) \in (0, 1)$ is defined in (1.4).

Proof of proposition 2.1: In the following, we introduce a covering technique to derive proposition 2.1 from corollary 2.6. Note that the estimate on the space-time L^2 norm of $\partial_s w$ was already proved in (ii) of corollary 2.6. Thus, we focus on the space-time L^{p+1} norm of w and L^2 norm ∇w . For that, we introduce a new covering technique to extend the estimate of any known space-time L^q norm of w , $\partial_s w$ or ∇w from $B_{1/2}$ to the whole unit ball. Note that, we follow the covering method of Merle and Zaag in [9]. Therefore, we don't give all the details. Here, we strongly need the following local space-time generalization of the notion of characteristic point: a point $(x_0, T_0) \in \bar{D}_u$ is δ_0 -non characteristic with respect to t where $\delta_0 \in (0, 1)$ if

$$u \text{ is defined on } \mathcal{D}_{x_0, T_0, t, \delta_0} \quad (2.33)$$

where

$$\mathcal{D}_{x_0, T_0, t, \delta_0} = \{(\xi, \tau) \neq (x_0, T_0) \mid t \leq \tau \leq T_0 - \delta_0 |\xi - x_0|\}. \quad (2.34)$$

We also define $\mathcal{S}_{x_0, T_0, t, \delta_0}$ the slice of $\mathcal{D}_{x_0, T_0, t, \delta_0}$ between $\tau = t$ and $\tau = T_0 - e^{-10}(T_0 - t)$ by

$$\mathcal{S}_{x_0, T_0, t, \delta_0} = \{(\xi, \tau) \mid t \leq \tau \leq T_0 - e^{-10}(T_0 - t), |\xi - x_0| \leq \frac{T_0 - \tau}{\delta_0}\}. \quad (2.35)$$

In fact, we find it easier to work in the $u(x, t)$ setting, in order to respect the geometry of the blow-up set. We claim the following:

LEMMA 2.7 (Covering technique) Consider $\kappa \geq 0$, $q \geq 1$ and $f \in L_{loc}^q(D_u)$. Then, for all $x_0 \in \mathbb{R}^N$, $T_0 \leq T(x_0)$ and $t_1 \leq T_0$ such that $\mathcal{D}_{x_0, T_0, t_1, \delta_0} \subset D_u$ for some $\delta_0 \in (0, 1)$, we have:

(i) For any x such that $|x - x_0| \leq \frac{T_0 - t_1}{\delta_0}$, f is defined on $\mathcal{S}_{x, T^*(x), t_1, 1}$.

(ii)
$$\sup_{\{x \mid |x - x_0| \leq \frac{T_0 - t_1}{\delta_0}\}} \int_{\mathcal{S}_{x, T^*(x), t_1, 1}} (T^*(x) - t)^\kappa |f(\xi, t)|^q d\xi dt < +\infty.$$

(iii)
$$\begin{aligned} & \sup_{\{x \mid |x - x_0| \leq \frac{T_0 - t_1}{\delta_0}\}} \int_{t_1}^{t_2(x)} (T^*(x) - t)^\kappa \int_{B(x, T^*(x) - t)} |f(\xi, t)|^q d\xi dt \\ & \leq C(\delta_0, \kappa) \sup_{\{x \mid |x - x_0| \leq \frac{T_0 - t_1}{\delta_0}\}} \int_{t_1}^{t_2(x)} (T^*(x) - t)^\kappa \int_{B(x, \frac{T^*(x) - t}{2})} |f(\xi, t)|^q d\xi dt, \end{aligned}$$

where $t_2(x) = T^*(x) - e^{-10}(T^*(x) - t_1)$ and where $T^*(x)$ is defined in (2.2).

Remarks 2.2.

- The point $(x, T^*(x))$ is on the lateral boundary of $\mathcal{D}_{x_0, T_0, t, \delta_0}$.
- Note that the supremum is taken over the basis of $\mathcal{D}_{x_0, T_0, t_1, \delta_0}$.

Proof of lemma 2.7: See Appendix A. ■

In the following, we give the proof only for the space-time L^{p+1} norm of w , since the space-time L^2 norm of ∇w follows exactly in the same way. Consider x_0 is a non characteristic point and $T_0 \in (t_0(x_0), T(x_0)]$, where $t_0(x_0)$ is defined in (2.28). Consider $s \geq -\log(T_0 - t_0(x_0))$, consider then $x \in \mathbb{R}^N$ such that, $|x - x_0| \leq \frac{e^{-s}}{\delta_0(x_0)}$. If $w = w_{x, T_0}$, then we write from the self-similar change of variables (1.7)

$$\int_s^{s+10} \int_{B(0, \rho)} |w_{x, T^*(x)}(y, \tau)|^{p+1} dy d\tau = \int_{t_1}^{t_2(x)} \int_{B(x, \rho(T^*(x) - t))} |u(\xi, t)|^{p+1} d\xi dt, \quad (2.36)$$

where $t_1 = t_1(s, T_0) = T_0 - e^{-s}$, $t_2(x) = t_2(x, s, T_0) = T^*(x) - e^{-10}(T^*(x) - t_1)$, with $\rho = 1$ or $\frac{1}{2}$. Note that $\mathcal{D}_{x_0, T_0, t_1, \delta_0(x_0)} \subset \mathcal{D}_{x_0, T(x_0), 0, \delta_0(x_0)} \subset \mathcal{D}_u$ by definition (1.4). Therefore, lemma 2.7 applies with $f = |u|^{p+1}$ (which is $L_{loc}^{p+1}(\mathcal{D}_u)$ from the solution of the Cauchy problem) and we write from iii) of lemma 2.7

$$\begin{aligned} & \sup_{\{x \mid |x - x_0| \leq \frac{e^{-s}}{\delta_0}\}} \left(\int_s^{s+10} \int_B |w_{x, T^*(x)}(y, \tau)|^{p+1} dy d\tau \right) \\ & \leq C(\delta_0) \sup_{\{x \mid |x - x_0| \leq \frac{e^{-s}}{\delta_0}\}} \left(\int_s^{s+10} \int_{B_{\frac{1}{2}}} |w_{x, T^*(x)}(y, \tau)|^{p+1} dy d\tau \right). \end{aligned}$$

Since the right-hand side is bounded by corollary 2.6, the same holds for the left-hand side (use in particular the Remark 2.2). This concludes the proof of proposition 2.1. \blacksquare

3 Boundedness of the solution in similarity variables

This section is divided in two parts:

- We first state a general version of theorem 1.1, uniform for x near x_0 and prove it. Then, we give a blow-up criterion for equation (1.8) based on the Lyapunov functional.
- We prove theorem 1.2.

3.1 A Lyapunov functional

Consider u a solution of (1.1) with blow-up graph $\Gamma : \{x \mapsto T(x)\}$ and x_0 is a non characteristic point. Consider $T_0 \in (t_0(x_0), T(x_0)]$, where $t_0(x_0)$ is defined in (2.28), for all $x \in \mathbb{R}^N$ is such that $|x - x_0| \leq \frac{T_0 - t_0(x_0)}{\delta_0(x_0)}$, where $\delta_0(x_0)$ is defined in (1.4), then we write w instead of $w_{x, T^*(x)}$ defined in (1.7) with $T^*(x)$ given in (2.2). We aim at proving that the functional H defined in (1.13) is a Lyapunov functional for equation (1.8), provided that s is large enough.

LEMMA 3.1 *For all $s \geq -\log(T^*(x) - t_0(x_0))$, we have the following inequality,*

$$\frac{d}{ds}(E(w)) \leq - \int_{\partial B} (\partial_s w)^2(\sigma, s) d\sigma + \Sigma(s), \quad (3.1)$$

where $\Sigma(s)$ satisfies

$$\Sigma(s) \leq C e^{-2\gamma s} + C e^{-2\gamma s} \int_B \left(w^2 + |\nabla w|^2 + (\partial_s w)^2 + |w|^{p+1} \right) dy. \quad (3.2)$$

Proof: Multiplying (1.8) by $\partial_s w$, and integrating over the ball B , we obtain, for all $s \geq -\log(T^*(x) - t_0(x_0))$, (recall from [8] that in the case where, $(f, g) \equiv (0, 0)$, we have $\frac{d}{ds} E_0(w) = - \int_{\partial B} (\partial_s w)^2(\sigma, s) d\sigma$).

$$\begin{aligned}
\frac{d}{ds}(E_0(w) + I_0(w)) &= - \int_{\partial B} (\partial_s w)^2(\sigma, s) d\sigma + \underbrace{\frac{2(p+1)}{p-1} e^{-\frac{2(p+1)s}{p-1}} \int_B F\left(e^{\frac{2s}{p-1}} w\right) dy}_{I_1} \\
&\quad + \underbrace{\frac{2}{p-1} e^{-\frac{2ps}{p-1}} \int_B f\left(e^{\frac{2s}{p-1}} w\right) w dy}_{I_2} \\
&\quad + \underbrace{e^{-\frac{2ps}{p-1}} \int_B g\left(e^{\frac{(p+1)s}{p-1}} (\partial_s w + y \cdot \nabla w + \frac{2}{p-1} w)\right) \partial_s w dy}_{I_3}. \quad (3.3)
\end{aligned}$$

By exploiting the fact that $|F(x)| + |xf(x)| \leq C(1 + |x|^{p+1})$, we deduce that for all $s \geq -\log(T^*(x) - t_0(x_0))$,

$$|I_1| + |I_2| \leq C e^{-2\gamma s} + C e^{-2\gamma s} \int_B |w|^{p+1} dy. \quad (3.4)$$

Note that by combining the inequality $|g(x)| \leq M(1 + |x|)$ and the fact that $-\log(T^*(x) - t_0(x_0)) \geq 0$, we obtain

$$|I_3| \leq C e^{-2\gamma s} + C e^{-2\gamma s} \int_B \left((\partial_s w)^2 + w^2 + |\nabla w|^2 \right) dy. \quad (3.5)$$

Then, by using (3.3), (3.4) and (3.5), we have the estimates (3.1) and (3.2) for all $s \geq -\log(T^*(x) - t_0(x_0))$. This concludes the proof of lemma 3.1. \blacksquare

With lemma 3.1 and proposition 2.1 we are in a position to prove theorem 1.1'.

THEOREM 1.1' (*Existence of a Lyapunov functional for equation (1.8)*)

Consider u a solution of (1.1) with blow-up graph $\Gamma : \{x \mapsto T(x)\}$ and x_0 is a non characteristic point. Then there exists $t_0(x_0) \in [0, T(x_0))$ such that, for all $T_0 \in (t_0(x_0), T_0(x_0)]$, for all $s \geq -\log(T_0 - t_0(x_0))$ and $x \in \mathbb{R}^N$, where $|x - x_0| \leq \frac{e^{-s}}{\delta_0(x_0)}$, we have

$$H(w(s+10)) - H(w(s)) \leq - \int_s^{s+10} \int_{\partial B} (\partial_s w)^2(\sigma, \tau) d\sigma d\tau, \quad (3.6)$$

where $w = w_{x, T^*(x)}$ and $T^*(x)$ is defined in (2.2).

Proof of theorem 1.1': Consider u is a solution of (1.1) with blow-up graph $\Gamma : \{x \mapsto T(x)\}$ and x_0 is a non characteristic point. We can apply proposition 2.1, if $\eta = \eta_0$ small enough, where we have $\frac{\eta(p+3)}{2} \leq \gamma$, for all $T_0 \in (t_0(x_0), T(x_0)]$, for all

$s \geq -\log(T_0 - t_0(x_0))$ and $x \in \mathbb{R}^N$, where $|x - x_0| \leq \frac{e^{-s}}{\delta_0(x_0)}$, we get

$$\begin{aligned} \int_s^{s+10} \int_B (\partial_s w)^2(y, \tau) dy d\tau + \int_s^{s+10} \int_B |w(y, \tau)|^{p+1} dy d\tau \\ + \int_s^{s+10} \int_B |\nabla w(y, \tau)|^2 dy d\tau \leq K_1 e^{\gamma s}. \end{aligned} \quad (3.7)$$

By combining (3.7) and the following Jensen's inequality

$$\int_B w^2 dy \leq C \int_B |w|^{p+1} dy + C,$$

we obtain

$$\int_s^{s+10} e^{-2\gamma\tau} \int_B \left((\partial_s w)^2 + w^2 + |\nabla w|^2 + |w|^{p+1} \right) dy d\tau \leq CK_1 e^{-\gamma s}. \quad (3.8)$$

Therefore, lemma 3.1 and (3.8), implies

$$E(w)(s+10) - E(w)(s) \leq - \int_s^{s+10} \int_{\partial B} (\partial_s w)^2(\sigma, \tau) d\sigma d\tau + CK_1 e^{-\gamma s}. \quad (3.9)$$

Then, we write

$$H(w)(s+10) - H(w)(s) \leq - \int_s^{s+10} \int_{\partial B} (\partial_s w)^2(\sigma, \tau) d\sigma d\tau + (CK_1 - \sigma(1 - e^{-10\gamma}))e^{-\gamma s},$$

where H is defined in (1.13).

We now choose σ large enough, so we have $CK_1 - \sigma(1 - e^{-10\gamma}) \leq 0$ and then (3.6) is a direct consequence of the above inequality. This concludes the proof of theorem 1.1'. ■

We now claim the following lemma:

LEMMA 3.2 *There exists $S_2 \geq S_1$ such that, if $H(w(s_2)) < 0$ for some $s_2 \geq \max(S_2, -\log(T^*(x) - t_0(x_0)))$, then w blows up in some finite time $S > s_2$.*

Proof: The argument is the same as in the corresponding part in [3]. ■

3.2 Boundedness of the solution in similarity variables

We prove theorem 1.2 here. Note that the lower bound follows from the finite speed of propagation and wellposedness in $H^1 \times L^2$. For a detailed argument in the similar case of equation (1.5), see lemma 3.1 (page 1136) in [9].

We define the following time:

$$t_1(x_0) = \max(T(x_0) - e^{-S_2}, 0). \quad (3.10)$$

Given some $T_0 \in (t_1(x_0), T(x_0)]$, for all $x \in \mathbb{R}^N$ is such that $|x - x_0| \leq \frac{T_0 - t_1(x_0)}{\delta_0(x_0)}$, where $\delta(x_0)$ is defined in (1.4), then we write w instead of $w_{x, T^*(x)}$ defined in (1.7) with $T^*(x)$ given in (2.2). We aim at bounding $\|(w, \partial_s w)(s)\|_{H^1 \times L^2(B)}$ for s large. As in [8], by combining theorem 1.1 and lemma 3.2 we get the following bounds:

COROLLARY 3.3 (*Bounds on E*) *For all $s \geq -\log(T^*(x) - t_1(x_0))$, it holds that*

$$\begin{aligned} -C &\leq E(w(s)) \leq M_0 \\ \int_s^{s+10} \int_{\partial B} \left(\partial_s w(\sigma, \tau) \right)^2 d\sigma d\tau &\leq M_0, \end{aligned} \quad (3.11)$$

where $M_0 = M_0(T_0, \|(u(t_0(x_0)), \partial_t u(t_0(x_0)))\|_{H^1 \times L^2(B(x_0, \frac{T(x_0) - t_0(x_0)}{\delta_0(x_0)}))})$, $C > 0$ and $\delta_0(x_0) \in (0, 1)$ is defined in (1.4).

We now claim the following corollary

COROLLARY 3.4 *For all $s \geq -\log(T^*(x) - t_1(x_0))$, we have*

$$\int_s^{s+10} \int_B \left(\partial_s w(y, \tau) - \lambda(\tau, s) w(y, \tau) \right)^2 dy d\tau \leq CM_0, \quad (3.12)$$

and (nonconcentration property) for all $b \in \mathbb{R}^N$ and $r_0 \in (0, 1)$ such that $B(b, r_0) \subset B(0, \frac{1}{\delta_0(x_0)})$,

$$\int_s^{s+\sqrt{r_0}} \int_{B(b, r_0)} \left(\partial_s w(y, \tau) - \lambda(\tau, s) w(y, \tau) \right)^2 dy d\tau \leq CM_0 r_0, \quad (3.13)$$

where $0 \leq \lambda(\tau, s) \leq C(\delta_0)$, for all $\tau \in [s, s + \sqrt{r_0}]$.

Proof: The argument is the same as in the corresponding part, see proposition 4.2 (page 1147) in [9]. ■

Starting from these bounds, the proof of theorem 1.2 is similar to the proof in [9] except for the treatment of the perturbation terms. In our opinion, handling these terms is straightforward in all the steps of the proof, except for the first step, where we bound the time averages of the $L^{p+1}(B)$ norm of w . For that reason, we only give that step and refer to [9] for the remaining steps in the proof of theorem 1.2. This is the step we prove here (In the following K_2, K_3 denotes a constant that depends only on C, M_0, ε_1 and ε_2 is an arbitrary positive number in $(0, 1)$).

PROPOSITION 3.5 (Control of the space-time L^{p+1} norm of w)

For all $s \geq -\log(T^*(x) - t_2(x_0))$, for some $t_2(x_0) \in [t_1(x_0), T(x_0))$, for all $\varepsilon_1 > 0$,

$$\begin{aligned} \int_s^{s+10} \int_B |w|^{p+1} dy d\tau &\leq \frac{K_3}{\varepsilon_1} + K_3 \varepsilon_1 \int_s^{s+10} \int_B |\nabla w|^2 dy d\tau \\ &+ C \int_B \left(|\partial_s w(y, s)|^2 + |\partial_s w(y, s+10)|^2 \right) dy \quad (3.14) \end{aligned}$$

Proof: By integrating the expression (1.14) of E in time between s and $s+10$, where $s \geq -\log(T^*(x) - t_1(x_0))$, we obtain:

$$\begin{aligned} \int_s^{s+10} E(\tau) d\tau &= \int_s^{s+10} \int_B \left(\frac{1}{2} (\partial_s w)^2 + \frac{p+1}{(p-1)^2} w^2 - \frac{1}{p+1} |w|^{p+1} \right) dy d\tau \\ &+ \frac{1}{2} \int_s^{s+10} \int_B \left(|\nabla w|^2 - (y \cdot \nabla w)^2 \right) dy d\tau \\ &- \int_s^{s+10} e^{-\frac{2(p+1)\tau}{p-1}} \int_B F(e^{\frac{2}{p-1}\tau} w) dy d\tau. \quad (3.15) \end{aligned}$$

By multiplying the equation (1.8) by w and integrating both in time and in space over $B \times [s, s+10]$, we obtain the following identity, after some integration by parts:

$$\begin{aligned} &\left[\int_B \left(w \partial_s w + \left(\frac{p+3}{2(p-1)} - N \right) w^2 \right) dy \right]_s^{s+10} = \int_s^{s+10} \int_B (\partial_s w)^2 dy d\tau \\ &- \int_s^{s+10} \int_B (|\nabla w|^2 - (y \cdot \nabla w)^2) dy d\tau - \frac{2p+2}{(p-1)^2} \int_s^{s+10} \int_B w^2 dy d\tau \\ &+ \int_s^{s+10} \int_B |w|^{p+1} dy d\tau + 2 \int_s^{s+10} \int_B \partial_s w (y \cdot \nabla w) dy d\tau - 2 \int_s^{s+10} \int_{\partial B} w \partial_s w d\sigma d\tau \\ &+ \int_s^{s+10} \int_B e^{-\frac{2p\tau}{p-1}} f\left(e^{\frac{2\tau}{p-1}} w\right) w dy d\tau \\ &+ \int_s^{s+10} \int_B e^{-\frac{2p\tau}{p-1}} g\left(e^{\frac{(p+1)\tau}{p-1}} \left(\partial_s w + y \cdot \nabla w + \frac{2}{p-1} w \right) \right) w dy d\tau. \quad (3.16) \end{aligned}$$

By combining the identities (3.15) and (3.16), we obtain

$$\begin{aligned}
\frac{(p-1)}{2(p+1)} \int_s^{s+10} \int_B |w|^{p+1} dy d\tau = & \\
& \frac{1}{2} \left[\int_B \left(w \partial_s w + \left(\frac{p+3}{2(p-1)} - N \right) w^2 \right) dy \right]_s^{s+10} - \int_s^{s+10} \int_B (\partial_s w)^2 dy d\tau \\
& + \int_s^{s+10} E(s) d\tau - \int_s^{s+10} \int_B \partial_s w (y \cdot \nabla w) dy d\tau + \int_s^{s+10} \int_{\partial B} w \partial_s w d\sigma d\tau \\
& - \underbrace{\frac{1}{2} \int_s^{s+10} \int_B e^{-\frac{2p\tau}{p-1}} g \left(e^{\frac{(p+1)\tau}{p-1}} \left(\partial_s w + y \cdot \nabla w + \frac{2}{p-1} w \right) w dy d\tau \right)}_{A_1} \\
& - \underbrace{\frac{1}{2} \int_s^{s+10} \int_B e^{-\frac{2p\tau}{p-1}} f \left(e^{\frac{2\tau}{p-1}} w \right) w dy d\tau}_{A_2} \\
& + \underbrace{\int_s^{s+10} e^{-\frac{2(p+1)\tau}{p-1}} \int_B F(e^{\frac{2}{p-1}\tau} w) dy d\tau}_{A_3}. \tag{3.17}
\end{aligned}$$

We claim that proposition 3.5 follows from the following lemma where we have the following estimates

LEMMA 3.6 *For all $s \geq -\log(T^*(x) - t_2(x_0))$, for some $t_2(x_0) \in [t_1(x_0), T(x_0))$, for all $\varepsilon_1 > 0$, for all $\varepsilon_2 > 0$,*

$$\int_s^{s+10} \int_B |\nabla w|^2 (1 - |y|^2) dy d\tau \leq K_2 + C \int_s^{s+10} \int_B |w|^{p+1} dy d\tau, \tag{3.18}$$

$$\sup_{\tau \in [s, s+10]} \int_B w^2(y, \tau) dy \leq \frac{K_2}{\varepsilon_2} + K_2 \varepsilon_2 \int_s^{s+10} \int_B |w|^{p+1} dy d\tau, \tag{3.19}$$

$$\begin{aligned}
\int_s^{s+10} \int_B |\partial_s w y \cdot \nabla w| dy d\tau & \leq \frac{K_2}{\varepsilon_1 \varepsilon_2} + K_2 \varepsilon_1 \int_s^{s+10} \int_B |\nabla w|^2 dy d\tau \\
& + K_2 \varepsilon_2 \int_s^{s+10} \int_B |w|^{p+1} dy d\tau, \tag{3.20}
\end{aligned}$$

$$\begin{aligned}
\int_s^{s+10} \int_{\partial B} |\partial_s w w| d\sigma d\tau & \leq \frac{K_2}{\varepsilon_1 \varepsilon_2} + K_2 \varepsilon_2 \int_s^{s+10} \int_B |w|^{p+1} dy d\tau \\
& + K_2 \varepsilon_1 \int_s^{s+10} \int_B |\nabla w|^2 dy d\tau, \tag{3.21}
\end{aligned}$$

$$\int_B |w \partial_s w| dy \leq \int_B (\partial_s w)^2 dy + \frac{K_2}{\varepsilon_2} + K_2 \varepsilon_2 \int_s^{s+10} \int_B |w|^{p+1} dy d\tau, \quad (3.22)$$

$$|A_1| \leq \frac{K_2}{\varepsilon_2} + K_2 \varepsilon_2 \int_s^{s+10} \int_B |w|^{p+1} dy d\tau + C e^{-2s} \int_s^{s+10} \int_B |\nabla w|^2 dy d\tau, \quad (3.23)$$

$$|A_2| + |A_3| \leq C + C e^{-2\gamma s} \int_s^{s+10} \int_B |w|^{p+1} dy d\tau. \quad (3.24)$$

Indeed, from (3.17) and this lemma, we deduce that

$$\begin{aligned} \int_s^{s+10} \int_B |w|^{p+1} dy d\tau &\leq \frac{K_2}{\varepsilon_1 \varepsilon_2} + (K_2 \varepsilon_2 + C e^{-2\gamma s}) \int_s^{s+10} \int_B |w|^{p+1} dy d\tau \\ &\quad + (K_2 \varepsilon_1 + C e^{-2s}) \int_s^{s+10} \int_B |\nabla w|^2 dy d\tau \\ &\quad + C \int_B \left(|\partial_s w(y, s)|^2 + |\partial_s w(y, s+10)|^2 \right) dy \end{aligned}$$

Now, we can use the fact that $s \geq -\log(T^*(x) - t_2(x_0)) \geq -\log(T(x_0) - t_2(x_0))$ and we prove easily that

$$e^{-2s} \leq (T(x_0) - t_2(x_0))^2 \quad \text{and} \quad e^{-2\gamma s} \leq (T(x_0) - t_2(x_0))^{2\gamma}.$$

Taking $t_2(x_0)$, where $T(x_0) - t_2(x_0)$ small enough, where we have

$$C(T(x_0) - t_2(x_0))^2 \leq K_2 \varepsilon_1 \quad \text{and} \quad (T(x_0) - t_2(x_0))^{2\gamma} \leq K_2 \varepsilon_2.$$

If we choose $\varepsilon_2 = \frac{1}{4K_2}$, we obtain (3.14).

It remains to prove lemma 3.6.

Proof of lemma 3.6: For the estimates (3.18), (3.19), (3.20), (3.21) and (3.22), we can adapt with no difficulty the proof given in the case of the wave equation treated in [7].

Now, we control the terms A_1 , A_2 and A_3 . Since $|g(x)| \leq M(1 + |x|)$, we have

$$\begin{aligned} |A_1| &\leq C e^{-s} \int_s^{s+10} \int_B (\partial_s w)^2 dy d\tau + C e^{-s} \int_s^{s+10} \int_B |y \cdot \nabla w| |w| dy d\tau \\ &\quad + C e^{-s} \int_s^{s+10} \int_B w^2 dy d\tau + C e^{-\frac{2ps}{p-1}} \int_s^{s+10} \int_B |w| dy d\tau. \end{aligned} \quad (3.25)$$

By using (3.12), we get

$$\int_s^{s+10} \int_B (\partial_s w)^2 dy d\tau \leq K_2 + C \int_s^{s+10} \int_B w^2 dy d\tau. \quad (3.26)$$

As usual, we write

$$C \int_s^{s+10} \int_B |w| dy d\tau \leq C + C \int_s^{s+10} \int_B w^2 dy d\tau. \quad (3.27)$$

Using the Cauchy-Schwarz inequality, we obtain

$$Ce^{-s} \int_s^{s+10} \int_B |w| |y \cdot \nabla w| dy d\tau \leq \int_s^{s+10} \int_B w^2 dy d\tau + Ce^{-2s} \int_s^{s+10} \int_B |\nabla w|^2 dy d\tau. \quad (3.28)$$

By combining the fact that $s \geq 0$ and the inequalities (3.19), (3.25), (3.26), (3.27) (3.28), we deduce

$$|A_1| \leq \frac{K_2}{\varepsilon_2} + K_2 \varepsilon_2 \int_s^{s+10} \int_B |w|^{p+1} dy d\tau + Ce^{-2s} \int_s^{s+10} \int_B |\nabla w|^2 dy d\tau. \quad (3.29)$$

Similarly, by exploiting the fact that $|F(x)| + |xf(x)| \leq C(1 + |x|^{p+1})$, we obtain

$$\begin{aligned} |A_2| + |A_3| &\leq C \int_s^{s+10} e^{-2\gamma\tau} d\tau + C \int_s^{s+10} e^{-2\gamma\tau} \int_B |w|^{p+1} dy d\tau \\ &\leq C + Ce^{-2\gamma s} \int_s^{s+10} \int_B |w|^{p+1} dy d\tau. \end{aligned} \quad (3.30)$$

This concludes the proof of lemma 3.6 and proposition 3.5 too. ■

Since the derivation of theorem 1.2 from proposition 3.5 is the same as in the non perturbed case treated in [9] (up to some very minor changes), this concludes the proof of theorem 1.2. ■

A Covering technique

In this section, we prove the covering result stated in lemma 2.7.

Proof of lemma 2.7:

(i) For all x such that $|x - x_0| \leq \frac{T_0 - t_1}{\delta_0}$, we easily have

$$T^*(x) - t_1 \leq T_0 - t_1, \quad t_2(x) \leq t_2(x_0) \quad \text{and} \quad B(x, T^*(x) - t_1) \subset B\left(x_0, \frac{T_0 - t_1}{\delta_0}\right), \quad (\text{A.1})$$

the basis of $\mathcal{D}_{x_0, T_0, t_1, \delta_0}$. Therefore

$$\mathcal{S}_{x, T^*(x), t_1, 1} \subset \mathcal{S}_{x_0, T_0, t_1, \delta_0} \subset \mathcal{D}_u \quad (\text{A.2})$$

and (i) follows.

(ii) Since $T^*(x) \leq T_0$ and $t_2(x) \leq t_2(x_0)$, we write from (A.2)

$$\int_{\mathcal{S}_{x, T^*(x), t_1, 1}} (T^*(x) - t)^\kappa |f(\xi, t)|^q d\xi dt \leq \int_{\mathcal{S}_{x_0, T_0, t_1, \delta_0}} (T_0 - t)^\kappa |f(\xi, t)|^q d\xi dt$$

which is finite since $f \in L_{loc}^q(D_u)$ and because there exists $0 < r_0 < 1$ such that $\mathcal{S}_{x_0, T_0, t_1, r_0 \delta_0} \subset D_u$ (this is true because in (A.2), D_u is open and $\mathcal{S}_{x_0, T_0, t_1, \delta_0}$ is closed). Thus, the supremum exists and (ii) is proved.

(iii) Consider x^* such that $|x^* - x_0| \leq \frac{T_0 - t}{\delta_0}$ and

$$\int_{\mathcal{S}_{x^*, T^*(x^*), t_1, 1}} (T^*(x^*) - t)^\kappa |f(\xi, t)|^q d\xi dt \geq \frac{1}{2} \sup_{|x - x_0| \leq \frac{T_0 - t}{\delta_0}} \int_{\mathcal{S}_{x, T^*(x), t_1, 1}} (T^*(x) - t)^\kappa |f(\xi, t)|^q d\xi dt.$$

It is enough to prove that

$$\begin{aligned} & \int_{\mathcal{S}_{x^*, T^*(x^*), t_1, 1}} (T^*(x^*) - t)^\kappa |f(\xi, t)|^q d\xi dt \\ & \leq C(\delta_0, \kappa) \sup_{|x - x_0| \leq \frac{T_0 - t_1}{\delta_0}} \int_{t_1}^{t_2(x)} (T^*(x) - t)^\kappa \int_{B(x, \frac{T^*(x) - t}{2})} |f(\xi, t)|^q d\xi dt \end{aligned} \quad (\text{A.3})$$

in order to conclude. In the following, we will prove (A.3).

Note that we can cover $\mathcal{S}_{x^*, T^*(x^*), t_1, 1}$ by $k(\delta_0)$ sets of the form $\mathcal{S}_{x_i, \widetilde{T}^*(x_i), t_1, \frac{1 - \delta_0}{2}}$ where $|x_i - x^*| \leq T^*(x^*) - t_1$ and

$$\widetilde{T}^*(x_i) = T^*(x^*) - \delta_0 |x_i - x^*|$$

is the later boundary in the backward cone of vertex $(x^*, T^*(x^*))$ and slape δ_0 . Indeed, this number does not change by scaling and is thus the same as the number when $T^*(x^*) - t_1 = 1$. In addition,

$$\begin{aligned} & |(T^*(x_i) - t_1) - (T^*(x^*) - t_1)| = |T^*(x_i) - T^*(x^*)| = \delta_0 ||x_i - x_0| - |x^* - x_0|| \\ & \leq \delta_0 |x_i - x^*| \leq \delta_0 (T^*(x^*) - t_1), \end{aligned}$$

hence

$$(1 - \delta_0)(T^*(x^*) - t_1) \leq T^*(x_i) - t_1 \leq (1 + \delta_0)(T^*(x^*) - t_1). \quad (\text{A.4})$$

Since $\widetilde{T}^*(x_i) \leq T^*(x_i)$, it follows that

$$\mathcal{S}_{x^*, T^*(x^*), t_1, 1} \subset \bigcup_{i=1}^{k(\delta_0)} \mathcal{S}_{x_i, \widetilde{T}^*(x_i), t_1, \frac{1 - \delta_0}{2}} \subset \bigcup_{i=1}^{k(\delta_0)} \left(\mathcal{S}_{x_i, T^*(x_i), t_1, \frac{1}{2}} \cap \{(\xi, \tau) \mid \tau \leq t_2(x^*)\} \right).$$

Moreover, for any $i = 1, \dots, k(\delta_0)$ and $t \in [t_1, \min(t_2(x^*), t_2(x_i))]$, we write

$$\begin{aligned} e^{-10}(T^*(x_i) - t_1) &= T^*(x_i) - t_2(x_i) \leq T^*(x_i) - t \leq T^*(x_i) - t_1, \\ e^{-10}(T^*(x^*) - t_1) &= T^*(x^*) - t_2(x^*) \leq T^*(x^*) - t \leq T^*(x^*) - t_1. \end{aligned}$$

Using (A.4), we see that

$$e^{-10}(1 - \delta_0)(T^*(x^*) - t) \leq T^*(x_i) - t \leq e^{10}(1 + \delta_0)(T^*(x^*) - t).$$

It follows then that

$$\begin{aligned} & \int_{\mathcal{S}_{x^*, T^*(x^*), t_1, 1}} (T^*(x^*) - t)^\kappa |f(\xi, t)|^q d\xi dt \\ & \leq \sum_{i=1}^{k(\delta_0)} \int_{\mathcal{S}_{x_i, T^*(x_i), t_1, \frac{1}{2}} \cap \{(\xi, t), t \leq t_2(x^*)\}} (T^*(x^*) - t)^\kappa |f(\xi, t)|^q d\xi dt \\ & \leq \sum_{i=1}^{k(\delta_0)} \int_{\mathcal{S}_{x_i, T^*(x_i), t_1, \frac{1}{2}}} \frac{e^{10\kappa}}{(1 - \delta_0)^\kappa} (T^*(x_i) - t)^\kappa |f(\xi, t)|^q d\xi dt \\ & \leq \frac{k(\delta_0)e^{10\kappa}}{(1 - \delta_0)^\kappa} \sup_{|x-x_0| \leq \frac{T_0-t_1}{\delta_0}} \int_{\mathcal{S}_{x, T^*(x), t_1, \frac{1}{2}}} (T^*(x) - t)^\kappa |f(\xi, t)|^q d\xi dt, \end{aligned}$$

where we used in the last line the fact that $x_i \in B(x^*, T^*(x^*) - t_1) \subset B(x_0, \frac{T_0-t_1}{\delta_0})$ by (A.1). This yields (iii) and concludes the proof of lemma 2.7. \blacksquare

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